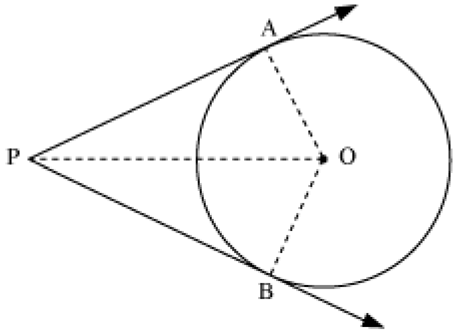
SECTION D

Question Numbers 29 to 34 carry 4 marks each.

29. Prove that the lengths of tangents drawn from an external point to a circle are equal.

Solution:



Let O be the centre of a circle.
Let PA and PB are two tangents drawn from a point P, lying outside the circle. Join OA, OB, and OP. We have to prove that PA = PB.
In ΔOPA and ΔOPB,
∠OAP = ∠OBP (Each equal to 90°)
(Since we know that a tangent at any point of a circle is perpendicular to the radius through the point of contact and hence, OA ⊥ PA and OB ⊥ PB).
OA = OB (Radii of the circle)
OP = PO (Common side)
Therefore, by RHS congruency criterion,
ΔOPA ≅ ΔOPB
∴ By CPCT,
PA = PB
Thus, the lengths of the two tangents drawn from an external point to a circle are equal.

30. A motor boat whose speed is 20 km/h is still water, takes 1 hour more to go 48 km upstream than to return downstream to the same spot. Find the speed of the stream.

Or

Find the roots of the equation $\frac{1}{x+4}-\frac{1}{x-7}=\frac{11}{30}$, $x \neq -4, 7$

Solution:
Let the speed of the stream be x km/h

Speed of the boat while going upstream = (20 – x) km/h
Speed of the boat while going downstream = (20 + x) km/h

Time taken for the upstream journey $= \dfrac{48 \text{ km}}{(20-x)\text{km/h}} = \dfrac{48}{20-x}\text{h}$

Time taken for downstream journey $= \dfrac{48 \text{ km}}{(20+x)\text{km/h}} = \dfrac{48}{20+x}\text{h}$

It is given that,
Time taken for the upstream journey = Time taken for the downstream journey + 1 hour

$$\Rightarrow \frac{48}{20-x} - \frac{48}{20+x} = 1$$

$$\Rightarrow \frac{960+48x-960+48x}{(20-x)(20+x)} = 1$$

$$\Rightarrow \frac{96x}{400-x^2} = 1$$

$$\Rightarrow 400-x^2 = 96x$$

$$\Rightarrow x^2 + 96x - 400 = 0$$

$$\Rightarrow x^2 + 100x - 4x - 400 = 0$$

$$\Rightarrow x(x+100) - 4(x+100) = 0$$

$$\Rightarrow (x+100)(x-4) = 0$$

$$\Rightarrow x+100 = 0 \text{ or } x-4 = 0$$

$$\Rightarrow x = -100 \text{ or } x = 4$$

$\therefore$ $x = 4$ [Since speed cannot be negative]

Thus, the speed of the stream is 4 km/h.

OR

$$\frac{1}{x+4}-\frac{1}{x-7}=\frac{11}{30}$$

$$\Rightarrow \frac{(x-7)-(x+4)}{(x+4)(x-7)}=\frac{11}{30}$$

$$\Rightarrow \frac{-11}{x^2-3x-28}=\frac{11}{30}$$

$$\Rightarrow x^2-3x-28=-30$$

$$\Rightarrow x^2-3x+2=0$$

$$\Rightarrow x^2-2x-x+2=0$$

$$\Rightarrow x(x-2)-1(x-2)=0$$

$$\Rightarrow (x-1)(x-2)=0$$

$$\Rightarrow x-1=0 \text{ or } x-2=0$$

$$\Rightarrow x=1 \text{ or } x=2$$

Hence, the roots of the given equation are 1 and 2.

31. If the sum of first 4 terms of an AP is 40 and that of first 14 terms is 280, find the sum of its first n terms.

Or

Find the sum of the first 30 positive integers divisible by 6.

Solution:

Let a and d respectively be the first term and the common difference of the given A.P.

The sum of first four terms

$$S_4=40$$

$$\Rightarrow \frac{4}{2}\{2a+(4-1)d\}=40$$

$$\Rightarrow 2a+3d=20 \qquad ...(1)$$

The sum of first 14 terms

$$S_{14}=280$$

$$\Rightarrow \frac{14}{2}\{2a+(14-1)d\}=280$$

$$\Rightarrow 2a+13d=40 \qquad ...(2)$$

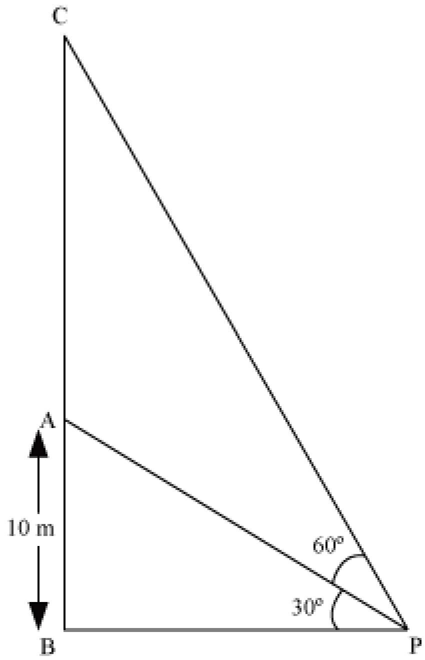
Subtracting equation (1) from equation (2),
$(2a + 13d) - (2a + 3d) = 40 - 20$
$\Rightarrow 10d = 20$
$\Rightarrow d = 2$
Substituting $d = 2$ in equation (1),
$2a + 3 \times 2 = 20$
$\Rightarrow 2a = 20 - 6 = 14$
$\Rightarrow a = \frac{14}{2} = 7$
$\therefore$ Sum of first n terms, $S_n = \frac{n}{2}\{2a + (n-1)d\}$
$= \frac{n}{2}\{2 \times 7 + (n-1) \times 2\}$
$= \frac{n}{2}(14 + 2n - 2)$
$\equiv \frac{n}{2}(2n + 12)$
$= \frac{n}{2} \times 2(n+6)$
$= n(n+6)$
$= n^2 + 6n$

OR

The first 30 integers divisible by 6 are 6, 12, 18 … 180.
Sum of first 30 integers
$= 6 + 12 + 18 + ... + 180$
$= \frac{30}{2}(6 + 180)$ $\quad \left[S_n = \frac{n}{2}(a + l)\right]$
$= 15 \times 186$
$= 2790$

32. From a point on the ground, the angles of elevation of the bottom and top of a transmission tower fixed at the top of a 10 m high building are 30° and 60° respectively. Find the height of the tower.

Solution:
The given information can be represented diagrammatically as:



In the figure P is the point of observation. AB is the building of height 10 m and AC is the transmission tower.

In ΔABP, $\frac{AB}{BP} = \tan 30°$

$\Rightarrow \frac{10\text{ m}}{BP} = \frac{1}{\sqrt{3}}$

$\Rightarrow BP = 10\sqrt{3}\text{ m}$

In ΔCBP, $\frac{CB}{BP} = \tan 60°$

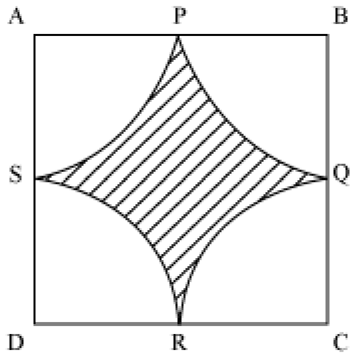
$\Rightarrow \frac{AB + AC}{10\sqrt{3}\text{ m}} = \sqrt{3}$

$\Rightarrow 10\text{ m} + AC = \sqrt{3} \times 10\sqrt{3}\text{ m} = 30\text{ m}$

$\Rightarrow AC = 30\text{ m} - 10\text{ m} = 20\text{ m}$

Thus, the height of the transmission tower is 20 m.

33. Find the area of the shaded region in Fig. 6, where arcs drawn with centres A, B, C and D intersect in pairs at mid-points P, Q, R and S of the sides AB, BC, CD and DA respectively of a square ABCD, where the length of each side of square is 14 cm. $\left[\text{Use } \pi = \frac{22}{7}\right]$



Solution:

Length of each side of the square = 14 cm

P, Q, R and S are the mid-points of sides AB, BC, CD and DA, respectively.

$\therefore$ Radius of each arc $= \frac{14}{2}$ cm $= 7$ cm

$\angle SAP = \angle PBQ = \angle QCR = \angle RDS = 90°$ [Angle of square]

$\therefore$ Area of sector APS = area of sector PBQ = area of sector QCR = area of sector SDR

Area of sector APS $= \frac{90°}{360°} \times \frac{22}{7} \times (7 \text{ cm})^2$

$= \frac{1}{4} \times \frac{22}{7} \times 49 \text{ cm}^2$

$= 38.5 \text{ cm}^2$

Area of the non-shaded region = 4 × area of sector APS = 4 × 38.5 cm^2 = 154 cm^2

Area of the square = $(\text{side})^2$ = $(14 \text{ cm})^2$ = 196 cm^2

Area of the shaded region = area of the square – area of the non-shaded region

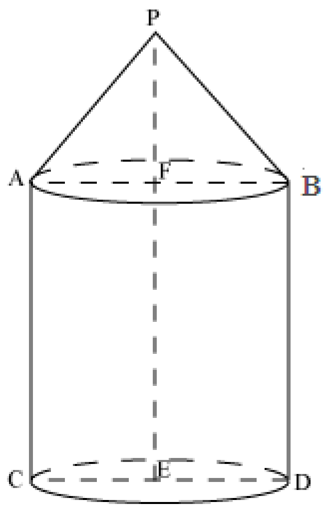
= 196 cm^2 – 154 cm^2 = 42 cm^2

Thus, the area of shaded region is 42 cm^2.

34. A toy is in the shape of a solid cylinder surmounted by a conical top. If the height and diameter of the cylindrical part are 21 cm and 40 cm respectively, and the height of cone is 15 cm, then find the total surface area of the toy. [π = 3.14, be taken]

Solution:

The given toy can be represented as,



Height of the cylindrical part (EF) = 21 cm

Diameter of the cylindrical part (CD = h_1) = 40 cm
$\therefore$ Radius of the cylindrical part (r_1) = 20 cm

Height of the conical part (PF = h_2) = 15 cm
Radius of the conical part (r_2) = r_1 = 20 cm

$r_1 = r_2 = 20$ cm = r (say)

Total surface area of the toy
= Curved surface area of the cylindrical part + Curved surface area of the conical part + area of the base of the cylindrical part

$$= 2\pi r_1 h_1 + \pi r_2\sqrt{r_2^2 + h_2^2} + \pi r_1^2 \qquad \left[l = \sqrt{r_2^2 + h_2^2}\right]$$

$$= 2\pi r h_1 + \pi r\sqrt{r^2 + h_2^2} + \pi r^2$$

$$= \pi r\left(2h_1 + \sqrt{r^2 + h_2^2} + r\right)$$

$$= 3.14 \times 20\left\{2 \times 21 + \sqrt{(20)^2 + (15)^2} + 20\right\} \text{ cm}^2$$

$$= 62.8\left(42 + \sqrt{625} + 20\right) \text{ cm}^2$$

$$= 62.8(42 + 25 + 20) \text{ cm}^2$$

$$= 62.8 \times 87 \text{ cm}^2$$

$$= 5463.6 \text{ cm}^2$$

Thus, the total surface area of the toy is 5463.6 cm^2.