PART – II : MATHEMATICS

SECTION – I (Single Correct Choice Type)

29. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A$ is

A) $\frac{1}{2}$ B) $\frac{\sqrt{3}}{2}$ C) 1 D) $\sqrt{3}$

ANSWER: D

30. Equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ is

A) $x + 2y - 2z = 0$ B) $3x + 2y - 2z = 0$ C) $x - 2y + z = 0$ D) $5x + 2y - 4z = 0$

ANSWER: C

31. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1}+\omega^{r_2}+\omega^{r_3}=0$ is

A) $\frac{1}{18}$ B) $\frac{1}{9}$ C) $\frac{2}{9}$ D) $\frac{1}{36}$

ANSWER: C

32. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i}-\hat{j}$, $4\hat{i}$, $3\hat{i}+3\hat{j}$ and $-3\hat{i}+2\hat{j}$ respectively. The quadrilateral PQRS must be a

A) parallelogram, which is neither a rhombus nor a rectangle
B) square
C) rectangle, but not a square
D) rhombus, but not a square

ANSWER: A

33. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ has exactly two distinct solutions, is

A) 0   B) $2^9 - 1$ C) 168   D) 2

ANSWER: A

34. The value of $\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t \ln(1+t)}{t^4+4} dt$ is

A) 0   B) $\frac{1}{12}$ C) $\frac{1}{24}$ D) $\frac{1}{64}$

ANSWER: B

35. Let p and q be real numbers such that $p \neq 0, p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is

A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$ B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$ D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

ANSWER: B

36. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then

A) a = b and c ≠ b   B) a = c and a ≠ b

C) a ≠ b and c ≠ b   D) a = b = c

ANSWER: D

SECTION – II (Multiple Correct Choice Type)

37. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be

A) $-\frac{1}{r}$ B) $\frac{1}{r}$ C) $\frac{2}{r}$ D) $-\frac{2}{r}$

ANSWER: C and D

38. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)

A) $-\left(2+\sqrt{3}\right)$ B) $1+\sqrt{3}$ C) $2+\sqrt{3}$ D) $4\sqrt{3}$

ANSWER: B

39. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t) z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(w) denotes the principal argument of a nonzero complex number w, then

A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

B) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$

C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

D) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

ANSWER: A and C and D

40. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ell n\ x + \int_0^x \sqrt{1+\sin t}\, dt$. Then which of the following statement(s) is (are) true ?

A) $f''(x)$ exists for all $x \in (0, \infty)$

B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

D) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$

ANSWER: B and C

41. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$ is (are)

A) $\frac{22}{7} - \pi$ B) $\frac{2}{105}$ C) 0 D) $\frac{71}{15} - \frac{3\pi}{2}$

ANSWER: A

SECTION – III (Paragraph Type)

Paragraph for Questions 42 to 44

Let p be an odd prime number and T_p be the following set of 2×2 matrices :

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, 2, \ldots, p-1\} \right\}$$

42. The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det (A) divisible by p is

A) $(p-1)^2$ B) $2(p-1)$ C) $(p-1)^2 + 1$ D) $2p - 1$

ANSWER: D

43. The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is

[**Note :** The trace of a matrix is the sum of its diagonal entries.]

A) $(p-1)(p^2 - p + 1)$ B) $p^3 - (p-1)^2$ C) $(p-1)^2$ D) $(p-1)(p^2 - 2)$

ANSWER: C

44. The number of A in T_p such that det (A) is not divisible by p is

A) $2p^2$ B) $p^3 - 5p$ C) $p^3 - 3p$ D) $p^3 - p^2$

ANSWER: D

Paragraph for Questions 45 to 46

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

45. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is

A) $2x - \sqrt{5}y - 20 = 0$ B) $2x - \sqrt{5}y + 4 = 0$

C) $3x - 4y + 8 = 0$ D) $4x - 3y + 4 = 0$

ANSWER: B

46. Equation of the circle with AB as its diameter is

A) $x^2 + y^2 - 12x + 24 = 0$ B) $x^2 + y^2 + 12x + 24 = 0$

C) $x^2 + y^2 + 24x - 12 = 0$ D) $x^2 + y^2 - 24x - 12 = 0$

ANSWER: A

SECTION – IV (Integer Type)

47. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for $n = 0, \pm 1, \pm 2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is

ANSWER: 3

48. The maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is

ANSWER: 2

49. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}}$ and $\vec{b} = \frac{2\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{14}}$, then the value of $\left(2\vec{a} + \vec{b}\right) \cdot \left[\left(\vec{a} \times \vec{b}\right) \times \left(\vec{a} - 2\vec{b}\right)\right]$ is

ANSWER: 5

50. The line 2x + y = 1 is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is

ANSWER: 2

51. If the distance between the plane Ax – 2y + z = d and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is

ANSWER: 6

52. For any real number x, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–10, 10] by

$$f(x) = \begin{cases} x - [x] & \text{if } [x] \text{ is odd,} \\ 1 + [x] - x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x \, dx$ is

ANSWER: 4

53. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to

ANSWER: 1

54. Let S_k, k = 1, 2, ..., 100, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} \left| \left(k^2 - 3k + 1\right) S_k \right|$ is

ANSWER: 3

55. The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations

$$(y + z) \cos 3\theta = (xyz) \sin 3\theta$$

$$x \sin 3\theta = \frac{2 \cos 3\theta}{y} + \frac{2 \sin 3\theta}{z}$$

$$(xyz) \sin 3\theta = (y + 2z) \cos 3\theta + y \sin 3\theta$$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is

ANSWER: 3

56. Let f be a real-valued differentiable function on **R** (the set of all real numbers) such that f(1) = 1. If the y-intercept of the tangent at any point P(x, y) on the curve y = f(x) is equal to the cube of the abscissa of P, then the value of f(-3) is equal to

ANSWER: 9